Separable Four-dimensional Harmonic Oscillators and Representations of the Poincaré Group

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Abstract

It is possible to construct representations of the Lorentz group using four-dimensional harmonic oscillators. This allows us to construct three-dimensional wave functions with the usual rotational symmetry for space-like coordinates and one-dimensional wave function for time-like coordinate. It is then possible to construct a representation of the Poincaré group for a massive particles having the $O(3)$ internal space-time symmetry in its rest frame. This oscillator can also be separated into two transverse components and the two-dimensional world of the longitudinal and time-like coordinates. The transverse components remain unchanged under Lorentz boosts, while it is possible to construct the squeeze representation of the $O(1,1)$ group in the space of the longitudinal and time-like coordinates. While the squeeze representation forms the basic language for squeezed states of light, it can be combined with the transverse components to form the representation of the Poincarè group for relativistic extended particles.

1 Introduction

Harmonic oscillators played the essential role in the development of quantum mechanics. From the mathematical point of view, the present form of quantum mechanics did not advance too far from the oscillator framework. Thus, the first relativistic wave function has to be the oscillator wave function. With this point in mind, Dirac in 1945 wrote down a normalizable four-dimensional wave function and attempted to construct representations of the Lorentz group [1], and started a history of the oscillators which can be Lorentz-boosted [2, 3].

Let us consider the Minkowskian space consisting of the three-dimensional space of (x, y, z) and one time variable t. Then the quantity $(x^2 + y^2 + z^2 - t^2)$ remains invariant under Lorentz transformations. On the other hand, $(x^2 + y^2 + z^2 + t^2)$ is not invariant. Thus, the exponential form

$$\exp\left(x^2 + y^2 + z^2 - t^2\right) \tag{1}$$

is a Lorentz-invariant quantity, while the form

$$\exp\left(x^2 + y^2 + z^2 + t^2\right) \tag{2}$$

is not. The Gaussian function of Eq.(2) is localized in the t variable. It was Dirac who wrote down this normalizable Gaussian form in 1945 [1]. In 1963, the author of this report was fortunate enough to hear directly from Prof. Dirac that the physics of special relativity is much richer than writing down Lorentz-invariant quantities. This could mean that we should study more systematically the above normalizable form under the influence of Lorentz boosts, and this study has led to the observation that Lorentz boosts are squeeze transformations [4, 5].

The exponential form of Eq.(1) is Lorentz-invariant but cannot be normalized. This aspect was noted by Feynman *et al.* in their 1971 paper [6]. In their paper, Feynman *et al.* tell us not to use Feynman diagrams but to use oscillator wave functions when we approach relativistic bound states. In this report, we shall use much of the formalism given in Ref. [6], but not their Lorentz-invariant wave functions which are not normalizable.

In this report, we discuss the oscillator representations of the $O(3,1)$ and explain why this representation is adequate for internal space-time symmetry of relativistic extended hadrons. This group has $O(1,1)$ as a subgroup which describes Lorentz boosts along a given direction. It is shown that Lorentz boosts are squeeze transformations. It is shown also that the infinite-dimensional unitary representation of this squeeze group constitutes the mathematical basis for squeezed states of light.

In Sec. 2, we discuss the three-parameter subgroups of the six-parameter Lorentz group which leaves the four-momentum of the particle invariant. These groups govern the internal space-time symmetries of relativistic particles, and they are called Wigner's little groups [7]. In Sec. 3, it is shown that the covariant harmonic oscillators constitute a representation of the Poincaré group for relativistic extended particles. If we add Lorentz boosts to the oscillator formalism, the symmetry group becomes non-compact, and its unitary representations become infinite-dimensional. In Sec. 4, we study an infinite-dimensional unitary representation for the harmonic oscillator formalism.

2   Subgroups of the Lorentz Group

The Poincaré group is the group of inhomogeneous Lorentz transformations, namely Lorentz transformations followed by space-time translations. This group is known as the fundamental space-time symmetry group for relativistic particles. This ten-parameter group has many different representations. For space-time symmetries, we study first Wigner's little groups. The little group is a maximal subgroup of the six-parameter Lorentz group whose transformations leave the four-momentum of a given particle invariant. The little group therefore governs the internal space-time symmetry of the particles. Massive particles in general are known as spin degrees

of freedom. Massless particles in general have the helicity and gauge degrees of freedom.

Let J_i be the three generators of the rotation group and K_i be the three boost generators. They then satisfy the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \qquad [J_i, K_j] = i\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -i\epsilon_{ijk}J_k. \qquad (3)$$

The three-dimensional rotation group is a subgroup of the Lorentz group. If a particle is at rest, we can rotate it without changing the four-momentum. Thus, the little group for massive particles is the three-parameter rotation group. If the particle is boosted, it gains a momentum along the boosted direction. If it is boosted along the z direction, the boost operator becomes

$$B_3(\eta) = \exp\left(-i\eta K_3\right). \qquad (4)$$

The little group is then generated by

$$J_i' = B_3(\eta)J_3\left(B_3(\eta)\right)^{-1}. \qquad (5)$$

These boosted $O(3)$ generators satisfy the same set of commutation relations as the set for $O(3)$.

Table I. Covariance of the energy-momentum relation, and covariance of the internal space-time symmetry groups. The quark model and the parton model are two different manifestations of the same covariant entity.

Massive, Slow	COVARIANCE	Massless, Fast
$E = p^2/2m$	Einstein's $E = mc^2$	$E = cp$
S_3	Wigner's Little Group	S_3
S_1, S_2		Gauge Trans.
Quarks	Covariant Harmonic Oscillators	Partons

If the parameter η becomes infinite, the particle becomes like a massless particle. If we go through the contraction procedure spelled out by Inonu and Wigner in

1953 [8], the $O(3)$-like little group becomes contracted to the $E(2)$-like little group for massless particles generated by J_3, N_1, and N_2 [9, 10],where

$$N_1 = K_1 - J_2, \qquad N_2 = K_2 + J_1. \tag{6}$$

These N generators are known to generate gauge transformations for massless particles [11, 12]. Gauge transformations for spin-1 photons are well known. As for massless spin-1/2 particles, neutrino polarizations are due to gauge invariance.

The transition from massive to massless particles is illustrated in the second row of Table I. In Sec. 3, we shall discuss how a massive particle with space-time extension be boosted from its rest frame to an infinite-momentum frame. This aspect is illustrated in the third row of Table I.

3   Covariant Harmonic Oscillators

If we construct a representation of the Lorentz group using normalizable harmonic oscillator wave functions, the result is the covariant harmonic oscillator formalism [3]. The formalism constitutes a representation of Wigner's O(3)-like little group for a massive particle with internal space-time structure. This oscillator formalism has been shown to be effective in explaining the basic phenomenological features of relativistic extended hadrons observed in high-energy laboratories. In particular, the formalism shows that the quark model and Feynman's parton picture [13] are two different manifestations of one relativistic entity [3, 14].

The essential feature of the covariant harmonic oscillator formalism is that Lorentz boosts are squeeze transformations [4]. In the light-cone coordinate system, the boost transformation expands one coordinate while contracting the other so as to preserve the product of these two coordinate remains constant. We shall show that the parton picture emerges from this squeeze effect.

The covariant harmonic oscillator formalism has been discussed exhaustively in the literature, and it is not necessary to give another full-fledged treatment in the present paper. We shall discuss here one of the most puzzling problems in high-energy physics, namely whether quarks are partons. It is now a well-accepted view that hadrons are bound states of quarks. This view is correct if the hadron is at rest or nearly at rest. On the other hand, it appears as a collection of partons when it moves with a speed very close to that of light. This is called Feynman's parton picture [13].

Let us consider a bound state of two particles. For convenience, we shall call the bound state the hadron, and call its constituents quarks. Then there is a Bohr-like radius measuring the space-like separation between the quarks. There is also a time-like separation between the quarks, and this variable becomes mixed with the longitudinal spatial separation as the hadron moves with a relativistic speed. There are no quantum excitations along the time-like direction. On the other hand, there is the time-energy uncertainty relation which allows quantum transitions. It is possible

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to accommodate these aspect within the framework of the present form of quantum mechanics. The uncertainty relation between the time and energy variables is the c-number relation, which does not allow excitations along the time-like coordinate. We shall see that the covariant harmonic oscillator formalism accommodates this narrow window in the present form of quantum mechanics.

Let us consider a hadron consisting of two quarks. If the space-time position of two quarks are specified by x_a and x_b respectively, the system can be described by the variables

$$X = (x_a + x_b)/2, \qquad x = (x_a - x_b)/2\sqrt{2}. \tag{7}$$

The four-vector X specifies where the hadron is located in space and time, while the variable x measures the space-time separation between the quarks. In the convention of Feynman *et al.* [6], the internal motion of the quarks bound by a harmonic oscillator potential of unit strength can be described by the Lorentz-invariant equation

$$\frac{1}{2}\left\{x_\mu^2 - \frac{\partial^2}{\partial x_\mu^2}\right\}\psi(x) = \lambda\psi(x). \tag{8}$$

We use here the space-favored metric: $x^\mu = (x, y, z, t)$.

It is possible to construct a representation of the Poincaré group from the solutions of the above differential equation [3]. If the hadron is at rest, the solution should take the form

$$\psi(x, y, z, t) = \phi(x, y, z)\left(\frac{1}{\pi}\right)^{1/4}\exp\left(-t^2/2\right), \tag{9}$$

where $\phi(x, y, z)$ is the wave function for the three-dimensional oscillator. If we use the spherical coordinate system, this wave function will carry appropriate angular momentum quantum numbers. Indeed, the above wave function constitutes a representation of Wigner's $O(3)$-like little group for a massive particle [3]. There are no time-like excitations, and this is consistent with our observation of the real world. It was Dirac who noted first this space-time asymmetry in quantum mechanics [15]. However, this asymmetry is quite consistent with the $O(3)$ symmetry of the little group for hadrons.

Since the three-dimensional oscillator differential equation is separable in both the spherical and Cartesian coordinate systems, the spherical form of $\phi(x, y, z)$ consists of Hermite polynomials of x, y, and z. If the Lorentz boost is made along the z direction, the x and y coordinates are not affected, and can be dropped from the wave function. The wave function of interest can be written as

$$\psi^n(z, t) = \left(\frac{1}{\pi}\right)^{1/4}\exp\left(-t^2/2\right)\phi_n(z), \tag{10}$$

with

$$\phi_n(z) = \left(\frac{1}{\pi n! 2^n}\right)^{1/2} H_n(z)\exp(-z^2/2), \tag{11}$$

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where $\psi^n(z)$ is for the n-th excited oscillator state. The full wave function $\psi^n(z,t)$ is

$$\psi_0^n(z,t) = \left(\frac{1}{\pi n!2^n}\right)^{1/2} H_n(z) \exp\left\{-\frac{1}{2}\left(z^2+t^2\right)\right\}. \tag{12}$$

The subscript 0 means that the wave function is for the hadron at rest. The above expression is not Lorentz-invariant, and its localization undergoes a Lorentz squeeze as the hadron moves along the z direction [3].

It is convenient to use the light-cone variables to describe Lorentz boosts. The light-cone coordinate variables are

$$u = (z+t)/\sqrt{2}, \qquad v = (z-t)/\sqrt{2}. \tag{13}$$

In terms of these variables, the Lorentz boost along the z direction,

$$\begin{pmatrix} z' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}, \tag{14}$$

takes the simple form

$$u' = e^\eta u, \qquad v' = e^{-\eta}v, \tag{15}$$

where η is the boost parameter and is $\tanh^{-1}(v/c)$.

The wave function of Eq.(12) can be written as

$$\psi_o^n(z,t) = \psi_0^n(z,t) = \left(\frac{1}{\pi n!2^n}\right)^{1/2} H_n\left((u+v)/\sqrt{2}\right) \exp\left\{-\frac{1}{2}(u^2+v^2)\right\}. \tag{16}$$

If the system is boosted, the wave function becomes

$$\psi_\eta^n(z,t) = \left(\frac{1}{\pi n!2^n}\right)^{1/2} H_n\left((e^{-\eta}u+e^\eta v)/\sqrt{2}\right) \times \exp\left\{-\frac{1}{2}\left(e^{-2\eta}u^2+e^{2\eta}v^2\right)\right\}. \tag{17}$$

In both Eqs. (16) and (17), the localization property of the wave function in the uv plane is determined by the Gaussian factor, and it is sufficient to study the ground state only for the essential feature of the boundary condition. The wave functions in Eq.(16) and Eq.(17) then respectively become

$$\psi^0(z,t) = \left(\frac{1}{\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}(u^2+v^2)\right\}. \tag{18}$$

If the system is boosted, the wave function becomes

$$\psi_\eta(z,t) = \left(\frac{1}{\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}\left(e^{-2\eta}u^2+e^{2\eta}v^2\right)\right\}. \tag{19}$$

We note here that the transition from Eq.(18) to Eq.(19) is a squeeze transformation. The wave function of Eq.(18) is distributed within a circular region in the uv plane, and thus in the zt plane. On the other hand, the wave function of Eq.(19) is distributed in an elliptic region.

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4 Unitary Infinite-dimensional Representation

Let us go back to Eq.(16) and Eq.(17). We are now interested in writing them in terms of the one-dimensional oscillator wave functions given in Eq.(11). After some standard calculations [3], we can write the squeezed wave function as

$$\psi_\eta^n(z,t) = \left(\frac{1}{\cosh\eta}\right)^{n+1} \sum_k \left(\frac{(n+k)!}{n!k!}\right)^{1/2} (\tanh\eta)^k \phi_{n+k}(z)\phi_n(t). \qquad (20)$$

If the parameter η becomes zero, this form becomes the rest-frame wave function of Eq.(16).

It is sometimes more convenient to use the parameter β defined as

$$\beta = \tanh\eta. \qquad (21)$$

This parameter is the speed of the hadron divided by the speed of light. In terms of this parameter, the expression of Eq.(20) can be written as

$$\psi_\eta^n(z,t) = \left(1 - \beta^2\right)^{(n+1)/2} \sum_k \left(\frac{(n+k)!}{n!k!}\right)^{1/2} \beta^k \phi_{n+k}(x)\phi_n(t). \qquad (22)$$

If we take the integral

$$\int |\psi_\eta^n(z,t)|^2 dz dt = \left(1 - \beta^2\right)^{n+1} \sum_k \left(\frac{(n+k)!}{n!k!}\right) \beta^{2k}. \qquad (23)$$

The sum in the above expression is the same as the binomial expansion of

$$\left(1 - \beta^2\right)^{-(n+1)}.$$

Thus, the right hand side of Eq.(23) is 1. The power series expansion of Eq.(20) reflects a well-known but hard-to-prove mathematical theorem that unitary representations of non-compact groups are infinite-dimensional. The Lorentz group is a non-compact group.

If $n = 0$, the above form becomes simplified to

$$\psi_\eta(z,t) = \left(1 - \beta^2\right)^{1/2} \sum_k \beta^k \phi_k(z)\phi_k(t). \qquad (24)$$

This is the power series expansion of Eq.(19). This relatively simple form is very useful in many other branches of physics.

It is well known that the mathematics of the Fock space in quantum field theory is that of harmonic oscillators. Among them, the coherent-state representation occupies a prominent place because it is the basic language for laser optics. Recently, two photon coherent states have been observed and the photon distribution is exactly

like that of the wave function given in Eq.(20). These coherent states are commonly called squeezed states. It is very difficult to see why the word "squeeze" has to be associated with the power series expansion given in Eq.(20) or Eq.(22). It is however quite clear from the expression of Eq.(19) that Gaussian distribution is squeezed. Thus, the above representation tells us how squeezed states are squeezed.

Next, let us briefly discuss the role of this infinite dimensional representation in understanding the density matrix. For simplicity, we shall work with the squeezed ground-state wave function. From the wave function of Eq.(24), we can construct the pure-state density matrix

$$\rho_\eta(z, t; z', t') = \psi_\eta(z, t)\psi_\eta(z', t'), \tag{25}$$

which satisfies the condition $\rho^2 = \rho$:

$$\rho_\eta(z, t; z', t') = \int \rho_\eta(z, t; z'', t'')\rho_\eta(z'', t''; z', t')dz''dt''. \tag{26}$$

However, there are at present no measurement theories which accommodate the time-separation variable t. Thus, we can take the trace of the ρ matrix with respect to the t variable. Then the resulting density matrix is

$$\rho_\eta(z, z') = \int \psi_\eta(z, t) \left\{\psi_\eta(z', t)\right\}^* dt = \left(1 - \beta^2\right) \sum_k \beta^{2k} \phi_k(z)\phi_k^*(z'). \tag{27}$$

It is of course possible to compute the above integral using the analytical expression given in Eq.(19). The result is

$$\rho_\eta(z, z') = \left(\frac{1}{\pi \cosh 2\eta}\right)^{1/2} \exp\left\{-\frac{1}{4}[(z + z')^2/\cosh 2\eta + (z - z')^2 \cosh 2\eta]\right\}. \tag{28}$$

This form of the density matrix satisfies the trace condition

$$\int \rho(z, z)dz = 1. \tag{29}$$

The trace of this density matrix is one, but the trace of ρ^2 is less than one, as

$$Tr\left(\rho^2\right) = \int \rho_\eta^n(z, z')\rho_\eta^n(z', z)dz'dz = \left(1 - \beta^2\right)^2 \sum_k \beta^{4k}, \tag{30}$$

which is less than one and is $1/\left(1 + \beta^2\right)$. This is due to the fact that we do not know how to deal with the time-like separation in the present formulation of quantum mechanics. Our knowledge is less than complete.

The standard way to measure this ignorance is to calculate the entropy defined as [16, 17]

$$S = -Tr\left(\rho \ln(\rho)\right). \tag{31}$$

If we pretend to know the distribution along the time-like direction and use the pure-state density matrix given in Eq.(25), then the entropy is zero. However, if

we do not know how to deal with the distribution along t, then we should use the density matrix of Eq.(27) to calculate the entropy, and the result is [18]

$$S = 2[(\cosh \eta)^2 \ln(\cosh \eta) - (\sinh \eta)^2 \ln(\sinh \eta)]. \qquad (32)$$

In terms of the velocity parameter, this expression can be written as

$$S = \frac{1}{1-\beta^2} \ln \frac{1}{1-\beta^2} - \frac{\beta}{1-\beta^2} \ln \frac{\beta}{1-\beta^2}. \qquad (33)$$

From this we can derive the hadronic temperature [19].

In this report, the time-separation variable t played the role of an unmeasurable variable. The use of an unmeasurable variable as a "shadow" coordinate is not new in physics and is of current interest [20]. Feynman's book on statistical mechanics contains the following paragraph [21].

When we solve a quantum-mechanical problem, what we really do is divide the universe into two parts - the system in which we are interested and the rest of the universe. We then usually act as if the system in which we are interested comprised the entire universe. To motivate the use of density matrices, let us see what happens when we include the part of the universe outside the system.

In the present paper, we have identified Feynman's rest of the universe as the time-separation coordinate in a relativistic two-body problem. Our ignorance about this coordinate leads to a density matrix for a non-pure state, and consequently to an increase of entropy.

References

[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A183**, 284 (1945).

[2] H. Yukawa, Phys. Rev. **91**, 415 (1953); M. Markov, Nuovo Cimento Suppl. **3**, 760 (1956); V. L. Ginzburgh and V. I. Man'ko, Nucl. Phys. **74**, 577 (1965); Y. S. Kim, M. E. Noz, and S. H. Oh, J. Math. Phys. **20**, 1341 (1979).

[3] Y. S. Kim and M. E. Noz, *Theory and Applications of the Poincaré Group* (Reidel, Dordrecht, 1986).

[4] Y. S. Kim and M. E. Noz, Phys. Rev. D **8**, 3521 (1973).

[5] Y. S. Kim and M. E. Noz, *Phase Space Picture of Quantum Mechanics* (World Scientific, Singapore, 1991).

[6] R. P. Feynman, M. Kislinger, and F. Ravndal, Phys. Rev. D **3**, 2706 (1971).

[7] E. P. Wigner, Ann. Math. **40**, 149 (1939).

[8] E. Inonu and E. P. Wigner, Proc. Natl. Acad. Sci. (U.S.) **39**, 510 (1953).

[9] H. Bacry and N. P. Chang, Ann. Phys. **47**, 407 (1968); S. Ferrara and C. Savoy, in *Supergravity 1981*, S. Ferrara and J. G. Taylor eds. (Cambridge Univ. Press, Cambridge, 1982), p. 151. See also P. Kwon and M. Villasante, J. Math. Phys. **29**, 560 (1988); *ibid.* **30**, 201 (1989).

[10] D. Han, Y. S. Kim, and D. Son, Phys. Lett. B **131**, 327 (1983). See also D. Han, Y. S. Kim, M. E. Noz, and D. Son, Am. J. Phys. **52**, 1037 (1984). Y. S. Kim and E. P. Wigner, J. Math. Phys. **28**, 1175 (1987) and **31**, 55 (1990).

[11] A. Janner and T. Janssen, Physica **53**, 1 (1971); *ibid.* **60**, 292 (1972).

[12] D. Han, Y. S. Kim, and D. Son, Phys. Rev. D **26**, 3717 (1982).

[13] R. P. Feynman, in *High Energy Collisions*, Proceedings of the Third International Conference, Stony Brook, New York, edited by C. N. Yang *et al.* (Gordon and Breach, New York, 1969); J. D. Bjorken and E. A. Paschos, Phys. Rev. **185** 1975 (1969).

[14] Y. S. Kim, Phys. Rev. Lett. **63**, 348 (1989).

[15] P. A. M. Dirac, Proc. Roy. Soc. (London) **A114**, 243 and 710 (1927).

[16] J. von Neumann, *Die mathematische Grundlagen der Quanten-mechanik* (Springer, Berlin, 1932). See also J. von Neumann, *Mathematical Foundation of Quantum Mechanics* (Princeton University, Princeton, 1955).

[17] E. P. Wigner and M. M. Yanase, Proc. National Academy of Sciences (U.S.A.) **49**, 910 (1963).

[18] Y. S. Kim and E. P. Wigner, Phys. Lett. A **147**, 343 (1990).

[19] D. Han, Y. S. Kim, and Marilyn E. Noz, Phys. Lett. A **144**, 111 (1990).

[20] H. Umezawa, H. Matsumoto, and M. Tachiki, *Thermo Field Dynamics and Condensed States* (North-Holland, Amsterdam, 1982); A. Mann and M. Revzen, Phys. Lett. **134A**, 273 (1989); A. K. Ekert and P. L. Knight, Am. J. Phys. **57**, 692 (1989); A. Mann, M. Revzen, and H. Umezawa, Phys. Lett. **139A**, 197 (1989); Y. S. Kim and M. Li, Phys. Lett. **139A**. 445 (1989); A. Mann, M. Revzen, H. Umezawa, Phys. Lett. **140A**, 475 (1989).

[21] R. P. Feynman, *Statistical Mechanics* (Benjamin/Cummings, Reading, MA, 1972); D. Han, Y. S. Kim, and M. E. Noz, Am. J. Phys. (to be published).